

BOUNDARY CONDITIONS IN THE FORM OF LOADING, ANALYZED WITH FINITE-DIFFERENCE METHOD

Katalin HARANGUS¹, András KAKUCS², Botond Pál GÁLFI³

¹ Transilvania University of Brasov, Faculty of Mechanical Engineering, e-mail: katalin@ms.sapientia.ro

² Sapientia-Hungarian University of Transylvania, Faculty of Technical and Human Sciences, e-mail: kakucs2@yahoo.com

³ Autoliv Inc., Brasov, e-mail: bgalfi@yahoo.com

Abstract—This paper presents a finite-difference computational method, which describes the plane state of stress of the orthotropic materials. By using a function of the displacement potential, the determination of boundary condition as load for the vertical side on the left is exemplified. The convex corner case is treated separately, when the corner displacement is free and when it is prevented or prescribed. When displacement is imposed in both directions, we analyze the situation in which only one side is leaning, and when both sides are embedded. Depending on the number of points inside, on the sides and the number of convex and concave corners, the number of points on the imaginary boundary and inside the domain under analysis can be established.

Keywords—displacement potential function, finite-difference method, orthotropic materials, stress analysis

I. INTRODUCTION

FOR the integration differential equation with partial derivatives which describe the plane state of displacement or stress of the anisotropic, orthotropic and isotropic materials, a computational method with finite difference can be applied [1]–[6].

The problem can be expressed in stress leading to Airy function [3], [4]. The second order partial derivatives of this describe the stress field. With the help of the stresses and with the equations of the material the specific strains, respectively displacement can be determined [7], [8].

The disadvantage of using the Airy function is that all the boundary conditions must be given in stresses, because the displacement cannot be expressed in a direct way.

By analogy with the Airy function a „potential function” of the displacement can be used, which makes the prescription of the mixed boundary conditions possible [3]–[4], [6]. The partial derivatives of this function give the displacement in the direction of the coordinate axes. The derivatives of the displacement, namely the derivatives of superior order of the function of the displacement give the specific strains, and through

the application of the material equations, these derivatives of superior order will lead to the stress field. This points to the fact that the description of the boundary conditions under the form of prescribed stresses by the load distribution on the boundary becomes possible, because there is a direct relation with differential equations between the displacements and stresses [3], [6]. These relations are estimated with finite differences.

The disadvantage of the method is that we can have body forces only in one direction.

II. FORMULATION WITH FINITE DIFFERENCE SOLUTION

Starting from the idea of Airy stress function, we suppose that in the case of orthotropic materials there is a function $\Psi(x, y)$. Its partial derivatives give the projections of the displacement. Using Hooke's law, we transcribe the equilibrium equations in specific strains (considering the body force in x -direction $\mathbf{f}_x = \mathbf{0}$), then with the geometrical equations, we obtain [3]:

$$\begin{aligned} \beta_1 \frac{\partial^4 \Psi}{\partial x^4} + \beta_2 \frac{\partial^4 \Psi}{\partial x^3 \partial y} + \beta_3 \frac{\partial^4 \Psi}{\partial x^2 \partial y^2} + \\ + \beta_4 \frac{\partial^4 \Psi}{\partial x \partial y^3} + \beta_5 \frac{\partial^4 \Psi}{\partial y^4} = \beta_0 \cdot \mathbf{f}_y, \end{aligned} \quad (1)$$

where β_i are the coefficients of the differential equation of Ψ , and $\mathbf{f}_y$ is the body force in y -direction.

The solution of this is the potential function sought by us.

If the partial derivatives of the (1) are replaced by centered finite differences, we achieve the molecules presented in Fig. 1., where $\mathbf{h}$, $\mathbf{k}$ is grid lengths in the x - and y -directions.

	$-2 \cdot h$	$-h$	0	h	$2 \cdot h$
$2 \cdot k$		$-\frac{\beta_4}{4 \cdot h \cdot k^3}$	$\frac{\beta_5}{k^4}$	$\frac{\beta_4}{4 \cdot h \cdot k^3}$	
k	$-\frac{\beta_2}{4 \cdot h^3 \cdot k}$	$\frac{\beta_2}{2 \cdot h^3 \cdot k} + \frac{\beta_3}{h^2 \cdot k^2} + \frac{\beta_4}{2 \cdot h \cdot k^3}$	$-2 \cdot \frac{\beta_3}{h^2 \cdot k^2} - 4 \cdot \frac{\beta_5}{k^4}$	$-\frac{\beta_2}{2 \cdot h^3 \cdot k} + \frac{\beta_3}{h^2 \cdot k^2} - \frac{\beta_4}{2 \cdot h \cdot k^3}$	$\frac{\beta_2}{4 \cdot h^3 \cdot k}$
0	$\frac{\beta_1}{h^4}$	$-4 \cdot \frac{\beta_1}{h^4} - 2 \cdot \frac{\beta_3}{h^2 \cdot k^2}$	$6 \cdot \frac{\beta_1}{h^4} + 4 \cdot \frac{\beta_3}{h^2 \cdot k^2} + 6 \cdot \frac{\beta_5}{k^4}$	$-4 \cdot \frac{\beta_1}{h^4} - 2 \cdot \frac{\beta_3}{h^2 \cdot k^2}$	$\frac{\beta_1}{h^4}$
$-k$	$\frac{\beta_2}{4 \cdot h^3 \cdot k}$	$-\frac{\beta_2}{2 \cdot h^3 \cdot k} + \frac{\beta_3}{h^2 \cdot k^2} - \frac{\beta_4}{2 \cdot h \cdot k^3}$	$-2 \cdot \frac{\beta_3}{h^2 \cdot k^2} - 4 \cdot \frac{\beta_5}{k^4}$	$\frac{\beta_2}{2 \cdot h^3 \cdot k} + \frac{\beta_3}{h^2 \cdot k^2} + \frac{\beta_4}{2 \cdot h \cdot k^3}$	$-\frac{\beta_2}{4 \cdot h^3 \cdot k}$
$-2 \cdot k$		$\frac{\beta_4}{4 \cdot h \cdot k^3}$	$\frac{\beta_5}{k^4}$	$-\frac{\beta_4}{4 \cdot h \cdot k^3}$	

Fig. 1. The approximation with finite-difference method of the (1).

Therefore, in the point of coordinates (x, y) , which is the point (i, j) of the grid for calculating finite differences, we can write the following equation:

$$\begin{aligned}
 & \frac{\beta_4}{4 \cdot h \cdot k^3} \cdot \Psi(i-1, j-2) + \frac{\beta_5}{k^4} \cdot \Psi(i, j-2) - \\
 & - \frac{\beta_4}{4 \cdot h \cdot k^3} \cdot \Psi(i+1, j-2) + \frac{\beta_2}{4 \cdot h^3 \cdot k} \cdot \Psi(i-2, j-1) - \\
 & - \left(\frac{\beta_2}{2 \cdot h^3 \cdot k} - \frac{\beta_3}{h^2 \cdot k^2} + \frac{\beta_4}{2 \cdot h \cdot k^3} \right) \cdot \Psi(i-1, j-1) - \\
 & - \left(2 \cdot \frac{\beta_3}{h^2 \cdot k^2} + 4 \cdot \frac{\beta_5}{k^4} \right) \cdot \Psi(i, j-1) + \\
 & + \left(\frac{\beta_2}{2 \cdot h^3 \cdot k} + \frac{\beta_3}{h^2 \cdot k^2} + \frac{\beta_4}{2 \cdot h \cdot k^3} \right) \cdot \Psi(i+1, j-1) - \\
 & - \frac{\beta_2}{4 \cdot h^3 \cdot k} \cdot \Psi(i+2, j-1) + \frac{\beta_1}{h^4} \cdot \Psi(i-2, j) - \\
 & - \left(4 \cdot \frac{\beta_1}{h^4} + 2 \cdot \frac{\beta_3}{h^2 \cdot k^2} \right) \cdot \Psi(i-1, j) + \\
 & + \left(6 \cdot \frac{\beta_1}{h^4} + 4 \cdot \frac{\beta_3}{h^2 \cdot k^2} + 6 \cdot \frac{\beta_5}{k^4} \right) \cdot \Psi(i, j) - \\
 & - \left(4 \cdot \frac{\beta_1}{h^4} + 2 \cdot \frac{\beta_3}{h^2 \cdot k^2} \right) \cdot \Psi(i+1, j) + \\
 & + \frac{\beta_1}{h^4} \cdot \Psi(i+2, j) - \frac{\beta_2}{4 \cdot h^3 \cdot k} \cdot \Psi(i-2, j+1) + \\
 & + \left(\frac{\beta_2}{2 \cdot h^3 \cdot k} + \frac{\beta_3}{h^2 \cdot k^2} + \frac{\beta_4}{2 \cdot h \cdot k^3} \right) \cdot \Psi(i-1, j+1) - \\
 & - \left(2 \cdot \frac{\beta_3}{h^2 \cdot k^2} + 4 \cdot \frac{\beta_5}{k^4} \right) \cdot \Psi(i, j+1) - \\
 & - \left(\frac{\beta_2}{2 \cdot h^3 \cdot k} - \frac{\beta_3}{h^2 \cdot k^2} + \frac{\beta_4}{2 \cdot h \cdot k^3} \right) \cdot \Psi(i+1, j+1) + \\
 & + \frac{\beta_2}{4 \cdot h^3 \cdot k} \cdot \Psi(i+2, j+1) - \\
 & - \frac{\beta_4}{4 \cdot h \cdot k^3} \cdot f(i-1, j+2) + \frac{\beta_5}{k^4} \cdot \Psi(i, j+2) + \\
 & - \frac{\beta_4}{4 \cdot h \cdot k^3} \cdot \Psi(i, j+2) = -f_y(i, j).
 \end{aligned} \tag{2}$$

In these equations the values of Ψ function appear, taken in the neighboring points, resulting in a system of equations to be solved in the $\Psi(i, j)$ nodal values.

For the boundary points we can't apply the molecules from Fig. 1., because the values of Ψ appear in (2) in some non-existing external nodes. These values also appear when we apply (2) for the nodes next to the boundary ones: these external nodes define a new imaginary boundary beyond the physical one, increasing the number of the unknowns to be determined (Fig. 2.).

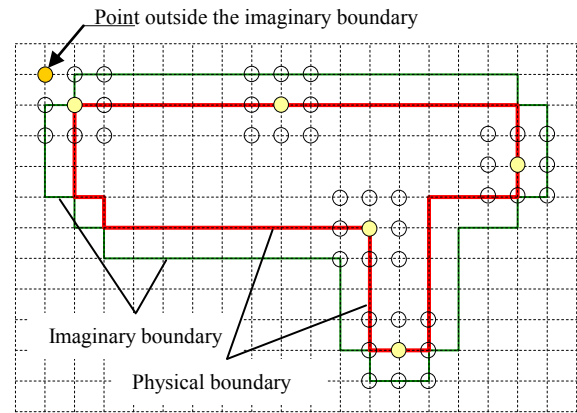


Fig. 2. The imaginary boundary that appears due to finite-difference approximation.

The system of equations can be solved only by writing boundary conditions: we will give these conditions in all boundary nodes, as prescribed displacements and/or loading forces.

III. THE BOUNDARY CONDITIONS IN THE FORM OF LOADING

Distributed stress that loads the boundary is defined by its projections according to x and y directions, noted as p_x and p_y . This stress is generally described by an arbitrary function. During the mesh, this function is replaced by a step function.

If we cut an element from the area around a boundary point, the stresses along the boundary must be in

equilibrium with the external loads. Therefore we can write the relations which equal the projections of the exterior load with the stresses along the boundary, and the boundary conditions are written by stresses with the

help of Ψ function derivatives, rewriting these derivatives with finite-differences (Fig. 3.).

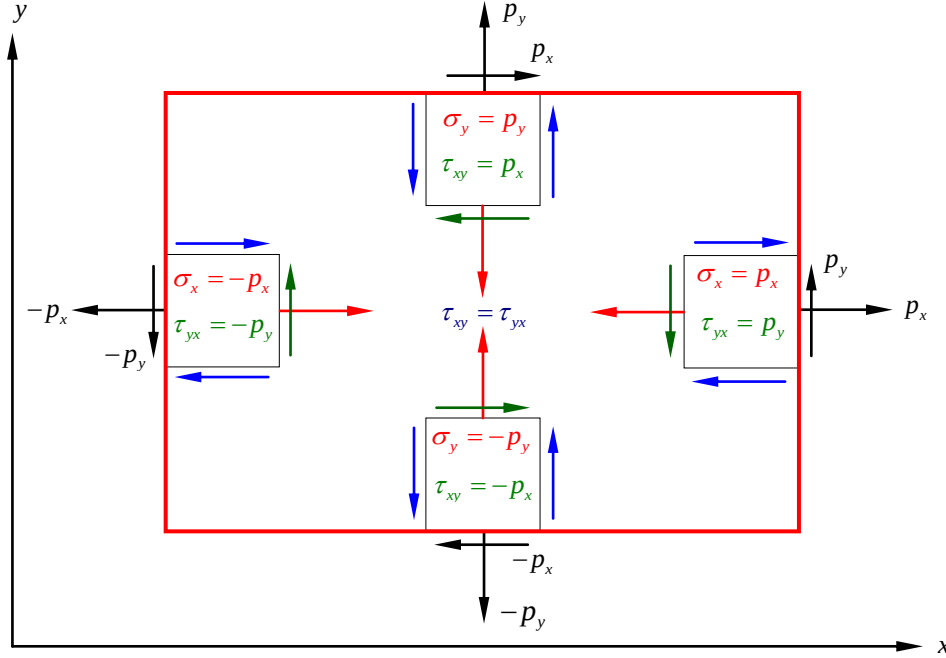


Fig. 3. Stresses along the boundary.

Relations between specific stresses and strains are given by the generalized Hooke's law, the specific deformations and displacements of geometrical equations and the displacements and Ψ function by (3).

$$\begin{aligned} u &= \alpha_1 \cdot \frac{\partial^2 \Psi}{\partial x^2} + \alpha_2 \cdot \frac{\partial^2 \Psi}{\partial x \cdot \partial y} + \alpha_3 \cdot \frac{\partial^2 \Psi}{\partial y^2}, \\ v &= \alpha_4 \cdot \frac{\partial^2 \Psi}{\partial x^2} + \alpha_5 \cdot \frac{\partial^2 \Psi}{\partial x \cdot \partial y} + \alpha_6 \cdot \frac{\partial^2 \Psi}{\partial y^2}, \end{aligned} \quad (3)$$

where, u , v are displacement components in the x - and y - directions, α_i are coefficients of the derivatives of Ψ .

Thus [4], we obtain the relation:

$$\begin{aligned} \sigma_x &= E_{11} \cdot \varepsilon_x + E_{12} \cdot \varepsilon_y + E_{13} \cdot \gamma_{xy} = \\ &= E_{11} \cdot \frac{\partial u}{\partial x} + E_{12} \cdot \frac{\partial v}{\partial y} + E_{13} \cdot \frac{\partial u}{\partial y} + E_{13} \cdot \frac{\partial v}{\partial x} = \\ &= E_{11} \cdot \left(\alpha_1 \cdot \frac{\partial^3 \Psi}{\partial x^3} + \alpha_2 \cdot \frac{\partial^3 \Psi}{\partial x^2 \cdot \partial y} + \alpha_3 \cdot \frac{\partial^3 \Psi}{\partial x \cdot \partial y^2} \right) + \\ &+ E_{12} \cdot \left(\alpha_4 \cdot \frac{\partial^3 \Psi}{\partial x^2 \cdot \partial y} + \alpha_5 \cdot \frac{\partial^3 \Psi}{\partial x \cdot \partial y^2} + \alpha_6 \cdot \frac{\partial^3 \Psi}{\partial y^3} \right) + \\ &+ E_{13} \cdot \left(\alpha_1 \cdot \frac{\partial^3 \Psi}{\partial x^2 \cdot \partial y} + \alpha_2 \cdot \frac{\partial^3 \Psi}{\partial x \cdot \partial y^2} + \alpha_3 \cdot \frac{\partial^3 \Psi}{\partial y^3} \right) + \\ &+ E_{13} \cdot \left(\alpha_4 \cdot \frac{\partial^3 \Psi}{\partial x^3} + \alpha_5 \cdot \frac{\partial^3 \Psi}{\partial x^2 \cdot \partial y} + \alpha_6 \cdot \frac{\partial^3 \Psi}{\partial x \cdot \partial y^2} \right) = \end{aligned} \quad (4)$$

$$\begin{aligned} &= (E_{11} \cdot \alpha_1 + E_{13} \cdot \alpha_4) \cdot \frac{\partial^3 \Psi}{\partial x^3} + \\ &+ (E_{11} \cdot \alpha_2 + E_{12} \cdot \alpha_4 + E_{13} \cdot \alpha_1 + E_{13} \cdot \alpha_5) \cdot \frac{\partial^3 \Psi}{\partial x^2 \cdot \partial y} + \\ &+ (E_{11} \cdot \alpha_3 + E_{12} \cdot \alpha_5 + E_{13} \cdot \alpha_2 + E_{13} \cdot \alpha_6) \cdot \frac{\partial^3 \Psi}{\partial x \cdot \partial y^2} + \\ &+ (E_{12} \cdot \alpha_6 + E_{13} \cdot \alpha_3) \cdot \frac{\partial^3 \Psi}{\partial y^3}, \end{aligned}$$

where σ_x is the normal stress, E_{ij} are the coefficients of the elasticity matrix.

The boundary conditions are written by stresses with the help of Ψ function derivatives, rewriting these derivatives with finite-differences.

We exemplify determining conditions of the boundary for the vertical side on the left, as follows:

$$\begin{aligned} & - \frac{c_4}{2 \cdot k^3} \cdot \Psi(j-2, i) - \\ & - \left(\frac{c_2}{2 \cdot h^2 \cdot k} + \frac{c_3}{2 \cdot h \cdot k^2} \right) \cdot \Psi(j-1, i-1) + \\ & + \left(\frac{c_2}{h^2 \cdot k} + \frac{c_4}{k^3} \right) \cdot \Psi(j-1, i) - \\ & - \left(\frac{c_2}{2 \cdot h^2 \cdot k} - \frac{c_3}{2 \cdot h \cdot k^2} \right) \cdot \Psi(j-1, i+1) + \end{aligned}$$

$$\begin{aligned}
 & + \frac{c_3}{hk^2} \cdot \Psi(j, i-1) - \frac{c_1}{h^3} \cdot \Psi(j, i) + \\
 & + \left(\frac{3 \cdot c_1}{h^3} - \frac{c_3}{hk^2} \right) \cdot \Psi(j, i+1) - \\
 & - \frac{3 \cdot c_1}{h^3} \cdot \Psi(j, i+2) + \frac{c_1}{h^3} \cdot \Psi(j, i+3) + \\
 & + \left(\frac{c_2}{2h^2k} - \frac{c_3}{2hk^2} \right) \cdot \Psi(j+1, i-1) - \\
 & - \left(\frac{c_2}{h^2k} + \frac{c_4}{k^3} \right) \cdot \Psi(j+1, i) + \\
 & + \left(\frac{c_2}{2h^2k} + \frac{c_3}{2hk^2} \right) \cdot \Psi(j+1, i+1) + \\
 & + \frac{c_4}{2k^3} \cdot \Psi(j+2, i) = -p_x,
 \end{aligned} \quad (5)$$

where,

$$\begin{aligned}
 c_1 &= E_{11} \cdot \alpha_1 + E_{13} \cdot \alpha_4, \\
 c_2 &= E_{11} \cdot \alpha_2 + E_{12} \cdot \alpha_4 + E_{13} \cdot \alpha_1 + E_{13} \cdot \alpha_5, \\
 c_3 &= E_{11} \cdot \alpha_3 + E_{12} \cdot \alpha_5 + E_{13} \cdot \alpha_2 + E_{13} \cdot \alpha_6, \\
 c_4 &= E_{12} \cdot \alpha_6 + E_{13} \cdot \alpha_3.
 \end{aligned} \quad (6)$$

We can observe that if we apply the molecule for a grid node positioned on the boundary, it will be based on three points that are on the imaginary boundary.

The concave corners will not raise issues or difficulties, however in the convex corners this scheme would include a point that does not belong to the imaginary boundary (Fig. 2.).

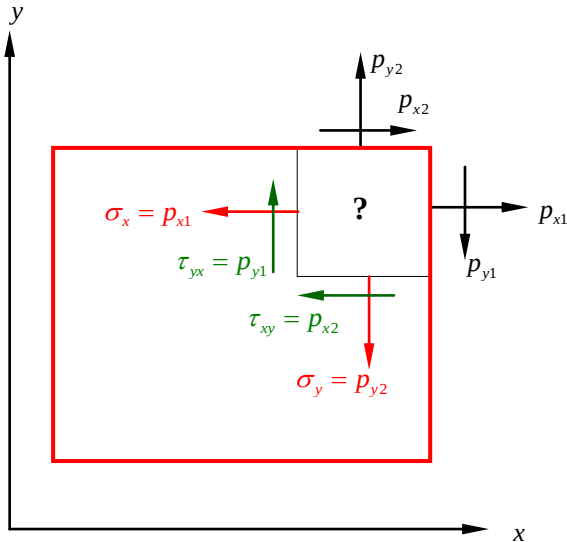


Fig. 4. Incompatible boundary conditions in convex corner.

Writing the boundary conditions as prescribed loads on the convex corners is an issue that should be considered separately. If we have tangential load (Fig. 4.) along the sides of boundaries the stress will be $\tau_{xy} = p_{x2}$ and $\tau_{yx} = p_{y1}$. When $p_{x2} \neq p_{y1}$ and/or their directions do not correspond to the principle of duality of tangential stresses, we have to redistribute the loads in these corners in order to solve this problem.

This redistribution is arbitrary. If the displacement of the corner is free in both directions, then the tangential stresses can be calculated with:

$$p'_{y1} = p'_{x2} = \frac{p_{y1} + p_{x2}}{2} \Rightarrow \tau_{xy} = \tau_{yx}, \quad (7.a)$$

and

$$p'_{x1} = p_{x1} + \left(p_{x2} - \frac{p_{y1} + p_{x2}}{2} \right) = \quad (7.b)$$

$$= p_{x1} + \frac{p_{x2} - p_{y1}}{2} \Rightarrow \sigma_x,$$

$$p'_{y2} = p_{y2} + \left(p_{y1} - \frac{p_{y1} + p_{x2}}{2} \right) = \quad (7.c)$$

$$= p_{y2} + \frac{p_{y1} - p_{x2}}{2} \Rightarrow \sigma_y.$$

This recalculation assures the principle of duality and preserves the resulted value of the load, $p'_{x1} + p'_{x2} = p_{x1} + p_{x2}$ and $p'_{y1} + p'_{y2} = p_{y1} + p_{y2}$.

If the displacement of the corner point is prevented or prescribed in one direction, then the tangential load in the grid node has to be eliminated.

It is considered that the supports are ideal, without friction, so in the direction of free displacement there is no tangential force. According to the principle of duality, in this case there can be no tangential force in the perpendicular direction to it (Fig. 5.). Prescribing null value of tangential stresses is the third equation which is written for those points.

In practice, there are two possibilities at corner points, where we imposed displacements in both directions:

1. If only at least one side is leaning: the third condition of boundary results because of lack of friction as null value of the tangential stress.

2. If both sides are embedded: the corner's turning point is blocked, so we can accept the null tangential tension ($\gamma = 0 \Rightarrow \tau = 0$).

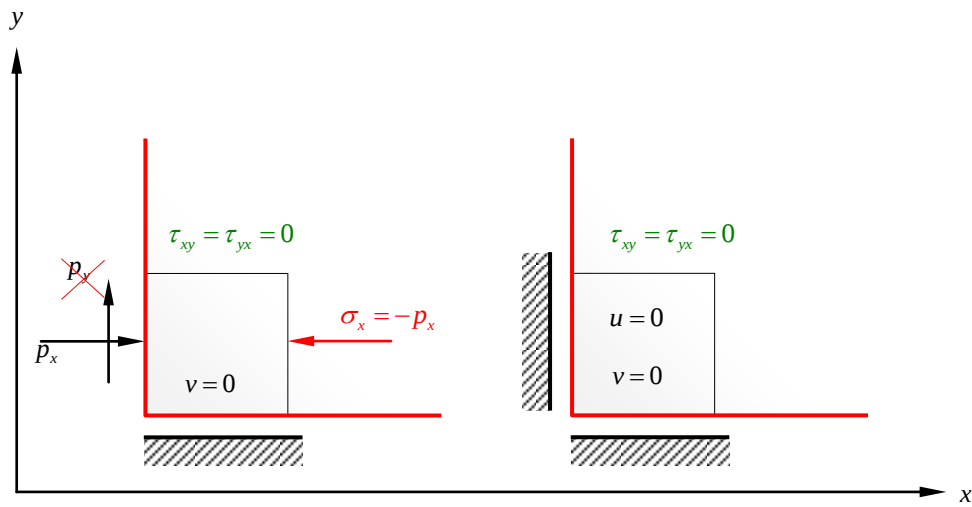


Fig. 5. Analyze of concave corner point.

Points of concave corners must be treated separately. Although there are no fictitious points, the normal of the contour is not defined, and we cannot separate one element of a rectangle the equilibrium of which we could analyze. The standard procedure would be rounding the corner points, that is to fill the corner with a triangular element and to analyze its equilibrium state (Fig. 6.).

Moving the point load on the hypotenuse of the corner:

$$\begin{aligned} F_x &= p_{x1} \cdot k + p_{x2} \cdot h, \\ F_y &= p_{y1} \cdot k + p_{y2} \cdot h. \end{aligned} \quad (8)$$

Equilibrium equations written for triangle lead to tensions on the legs. This problem has a unique solution because we cannot define the three stresses of the equilibrium equations written on the hypotenuse load projections. Therefore, the average tangential load is calculated according to (7.a), which gives the corresponding sign of the tangential stress. The difference between initial and average value is added to the normal load (7.b and c), the sum being equal to the normal stress.

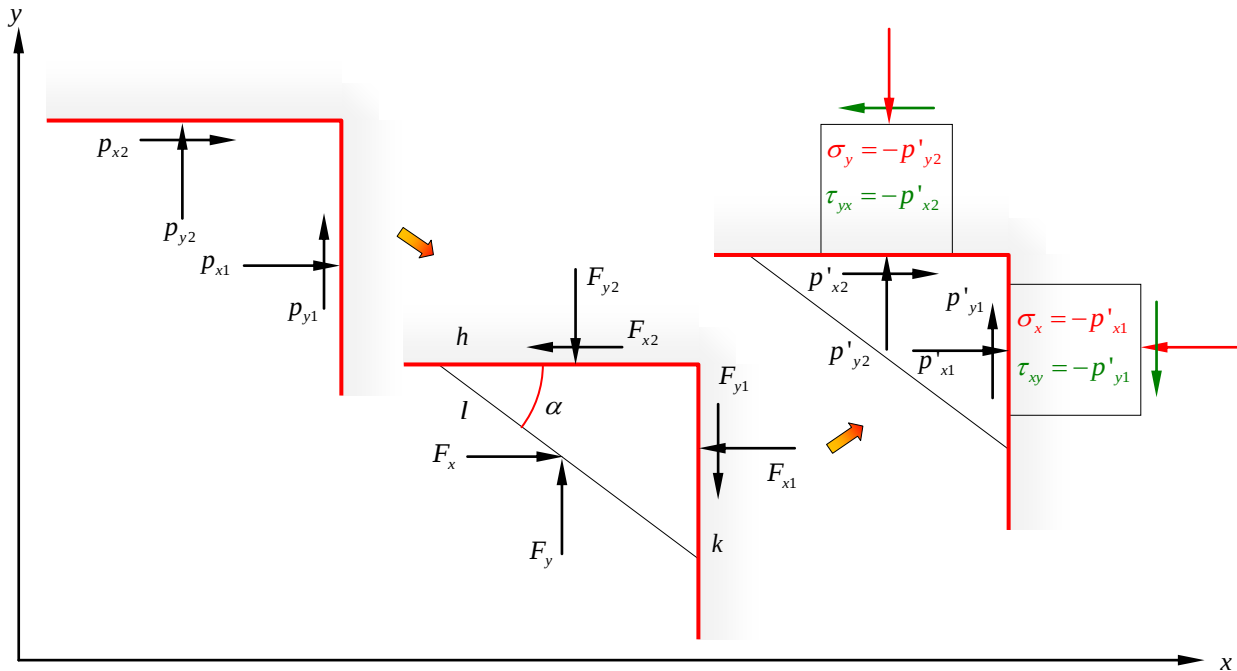


Fig. 6. Analyze of concave corner points.

IV. ESTABLISHING AND SOLVING THE SYSTEM OF EQUATION

Supposing that the studied domain is homogeneous, and comprises a number of interior points **b**, a number of points on the sides **e**, a number of convex corners **s**, and a number of concave corners **k**. Then the number of points on the imaginary boundary contour is $f = e + 2 \cdot s$, and the domain is covered in total of $p = b + 2 \cdot e + 3 \cdot s + k$ points.

Displacements in this domain are given by the function $\Psi(x, y)$. The values of this function in the grid points must be calculated by using the method of finite differences, a total of **p** points, which means a number of **p** equations, as follows:

- Equations **b**, which we write in the interior points by the scheme in Fig. 1. In the absence of volume forces the free term is zero.

- Equations $2 \cdot e$, which we write on the sides, corresponding to the boundary conditions valid on the sides.

- Equations $3 \cdot s$, which we write for the convex corners.

If we prescribe only the load in the convex corner, then for the centered difference approximation we apply the derivatives' approximations with the help of forward or backward differences, depending on the corner position. If the tangential components do not satisfy the principle of duality, then the loads are recalculated by (7). In this case the free term gives the load with the corresponding sign.

- Equations **k**, which we write for the concave corners.

In this way we obtain a system of linear equations, where the unknowns are the values of function Ψ at the grid points. The number of unknowns is equal to the number of the written equations, being equal to the number of grid points (including points on virtual boundary).

V. CONCLUSIONS

The defined problem was the solving of differential equation (1), satisfying the boundary conditions. The analyzed structure was replaced by a mesh grid of nodes, needed for writing the finite differences. The physical contour of the studied domain was replaced by a mesh with a broken line consisting of horizontal and vertical segments and boundary conditions as the prescribed loading led to some equivalence relations between loads and stresses. Each point on the sides introduces a point outside the domain, forming an imaginary contour parallel to the physical one. Two imaginary points belong to the convex corner, but none belong to the concave corners.

If the condition of boundary for the concave corner point is mixed or defines only the displacement projections, then the written equation for this point refers to the null value of the tangential stress. The concave corner does not introduce any new points on the imaginary boundary, so it is enough to write a single equation. In these points we define the sum of normal stresses in order to preserve symmetry (invariant of stress state).

For convex corners, if the displacement of the corner is free in both directions, the tangential stress can be calculated with the mean tangential load. If the displacement of the corner point is prevented, we must analyze the situation when one side is leaning, and when both sides are embedded.

It is noted, that if in a portion of boundary we do not define the load or the displacement, it means that on that portion it is loaded with a force distributed to zero intensity. So we have boundary condition in that case, too.

The denser the grid is, the more accurate the modeling of the load will be and the negative effects of the approximations made in the corner points will be more reduced. The disadvantage of this method is the fact that we can have body forces only in one direction.

REFERENCES

- [1] H.S. Paik, S.Y. Lee, and S.Y. Chang, "Finite Difference Stress Analysis of Anisotropic Three-dimensional Curved Bodies with Free Boundaries," *KSCE Journal of Civil Engineering*, vol. 8, no. 1, pp. 49-57, January 2004.
- [2] J.O. Dow, M.S. Jones, and S.A. Harwood, "A new approach to boundary modeling for finite difference applications in solid mechanics," *International Journal for Numerical Methods in Engineering*, vol. 30, pp. 99-113, July 1990.
- [3] K. Harangus, "Model of orthotropic materials with the finite-difference method," *Research and Science Today*, Academica Brancusi Publishing, 2(4), pp. 143-154, 2012.
- [4] K. Harangus, and A. Kakucs, "Finite difference modeling of orthotropic materials," *Acta Universitatis Sapientiae, Electrical and Mechanical Engineering*, vol. 4, 2012, submitted for publication.
- [5] M. Abdus S.A., M. Reaz A., S.R., M. Raisuddin K., and M. Wahhaj U., "A finite-difference scheme for mixed boundary-value problems of arbitrary-shaped elastic bodies," *Advances in Engineering Software*, 31(3), pp. 173-184, 2000.
- [6] S. Reaz A., M. Zubaer H., and M. Wahhaj E., "A general mathematical formulation for finite-difference solution of mixed-boundary-value problems of anisotropic materials," *Computer and Structures* vol. 83, pp. 35-51, August 2005.
- [7] A. Kakucs, *A finite-element method for the calculation of structures*. (A veges-elem módszer a szerkezetek számításában). Brasov: Editura Universitatii Transilvania, 2007.
- [8] S.P. Timoshenko, and J.N. Goodier, *Theory of elasticity*. 3rd ed. New York: McGraw-Hill Book Company, 1979.
- [9] Gy. Szeidl, "Kinematic Admissibility of Strains for Same Mixed Boundary Value Problems in the Dual System of Micropolar Theory of Elasticity," *Journal of Computational and Applied Mechanics*, 1(2), pp. 191-203, 2000.
- [10] Gy. Szeidl, "Boundary Integral Equations for Plane Problems in Terms of Stress Functions of Order One," *Journal of Computational and Applied Mechanics*, 2(2), pp. 237-261, 2001.